Proving e-voting mixnets in the CCSA model: zero-knowledge proofs and rewinding

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Laboratoire Méthodes Formelles

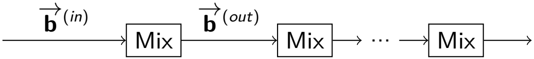
Electronic voting mixnets

Two kinds of tally

Homomorphic encryption

Mix networks + Decrypt

Principle



Network of *mix-servers*

Algorithm : Mixing

let `mixing` $\overrightarrow{\mathbf{b}}^{(in)}$ **=**

$\pi \xleftarrow{\$} \mathfrak{S}_N$;

[*do some stuff...*] ;

return $\overrightarrow{\mathbf{b}}^{(out)}$

Mix-server in a nutshell

Terelius & Wikström mixnet ([TW10], [Wik11])

Security properties for one mix-server

Permutation secrecy

Verifiability

Key ingredients needed

Commitment scheme

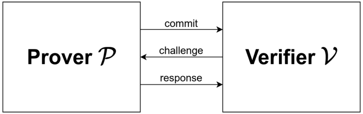
Zero-knowledge proofs

Zero-knowledge proofs - case of Σ-protocols

Principle

- Two agents: a prover $\mathcal{P}$ and a verifier $\mathcal{V}$
- Goal: prove that $(\underbrace{x}_{statement}, \underbrace{w}_{witness}) \in \mathcal{R}$
- Interactive proof: proof transcript

$$(\underbrace{p_0}_{commit}, \underbrace{c}_{challenge}, \underbrace{p_1}_{response})$$



Sigma-protocol

Main security properties

Special-Soundness

Zero-knowledge

Verifiability game

Cryptographic game — Mix-server verifiability.

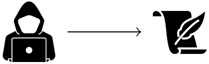
Context

Adversarial mix-server

Honest verifier $\mathcal{V}$

Game statement

Hypothesis



Proofs accepted by $\mathcal{V}$

$\Longrightarrow$

Conclusion

$$\mathrm{Dec}\ \overrightarrow{\mathbf{b}}^{(out)} = \mathrm{Dec}\left(M_\pi \cdot \overrightarrow{\mathbf{b}}^{(in)}\right)$$

Output plaintexts is a permutation of input

Computationally Complete Symbolic Attacker (CCSA) model

The SQUIRREL prover ([Bae+21])

- First introduce by Bana & Comon ([BC14]), high-order logic by Baelde, Koutsos & Lallemand ([BKL23])
- Main predicates: $\sim$ (indistinguishability) and $[\cdot]$ (globally (non-)negligible events)
- Interpretation of terms for a *fixed* random tape ρ: $[\![t]\!]_\rho$.
- In our case: work on trace properties
- Formulas ϕ are terms of type **bool**.

Two kinds of logic

Global logic

$[\phi] \ \tilde{\rightarrow} \ [\psi]$ means:
If $\mathrm{Pr}_{\rho\in\Omega}\left([\![\phi]\!]_\rho\right)$ is overwhelming
then $\mathrm{Pr}_{\rho\in\Omega}\left([\![\psi]\!]_\rho\right)$ is overwhelming.

Local logic

$[\phi \rightarrow \psi]$ means:
$\mathrm{Pr}_{\rho\in\Omega}\left([\![\phi \rightarrow \psi]\!]_\rho\right)$ is overwhelming.

Sketch of proof

Extraction of sealed matrix M

- *Witness extractor* — Rewinding
- *Collect enough witness* — Rewinding
- Reconstruction of sealed informations — Algebra

Is M a permutation matrix?

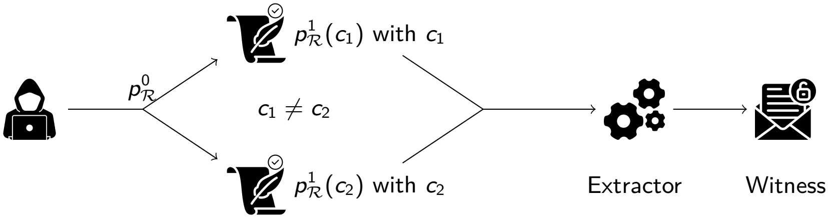
- *Witness extractor* — Rewinding
- Witness consistency — Cryptography
- Generalization of equations on witness to equations on matrix — Algebra
- Characterization of permutation matrix — Algebra

$\overrightarrow{\mathbf{b}}^{(out)} = \text{ReRand}\left(M \cdot \overrightarrow{\mathbf{b}}^{(in)}\right)$**?**

- *Another witness extractor* — Rewinding
- Consistency between the witness and the extracted matrix — Cryptography
- Generalization to the whole set of ciphertexts in/out pairs — Algebra

Special-Soundness

Statement



Axiomatization in the CCSA logic

L.SP:SpSo

$$\tilde{\exists}\ \mathtt{extract}_{\mathcal{R}}\ [\text{ptime}].\ \left[\bigwedge_{i \in \{1,2\}} \mathtt{verify}_{\mathcal{R}}(x, \underbrace{(p^0_{\mathcal{R}}, c_i, p^{1,(i)}_{\mathcal{R}})}_{\mathbf{p}^{(i)}_{\mathcal{R}}}) \wedge c_1 \neq c_2 \rightarrow (x, \mathtt{extract}_{\mathcal{R}}(x, \mathbf{p}^{(1)}_{\mathcal{R}}, \mathbf{p}^{(2)}_{\mathcal{R}})) \in \mathcal{R} \right]$$

Witness extraction algorithm

Algorithm : Witness extraction

Input: Adversary $\mathcal{A}$ producing sometimes a proof accepted by the verifier $\mathcal{V}$.

Run $p_0 \leftarrow \mathcal{A}(x)$;

repeat

    Choose $c_1 \leftarrow \mathcal{V}(1^\eta, x, p_0)$ then run $p_1 \leftarrow \mathcal{A}(x, p_0, c_1)$;

    Rewind $\mathcal{A}$;

    Choose $c_2 \leftarrow \mathcal{V}(1^\eta, x, p_0)$ then run $p_2 \leftarrow \mathcal{A}(x, p_0, c_2)$;

    Check if **true** $\leftarrow \mathcal{V}(x, \mathbf{p}_1)$ and **true** $\leftarrow \mathcal{V}(x, \mathbf{p}_2)$;

until $\mathbf{p}_1$ *and* $\mathbf{p}_2$ *are accepted by* $\mathcal{V}$ *and* $c_1 \neq c_2$;

return $w \leftarrow \texttt{extract}_{\mathcal{R}}(x, \mathbf{p}_1, \mathbf{p}_2)$;

where $\mathbf{p}_i := (p_0, c_i, p_i)$ for $i = 1, 2$.

First attempt

A first local hunch...

$$\textsc{L.Extract}\ \frac{\texttt{verify}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r_1))}{(x, \texttt{extract}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r_1), \mathbf{p}_{\mathcal{R}}(r_2))) \in \mathcal{R}}$$

where $\mathbf{p}_{\mathcal{R}} := \lambda r.(p_{\mathcal{R}}^{(0)}, r, p_{\mathcal{R}}^{(1)}(r))$ for some *fixed* $p_{\mathcal{R}}^{(0)}$.

First attempt

A first local hunch...

$$\textsc{L.Extract}\quad \frac{\texttt{verify}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r_1))}{(x, \texttt{extract}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r_1), \mathbf{p}_{\mathcal{R}}(r_2))) \in \mathcal{R}}$$

where $\mathbf{p}_{\mathcal{R}} := \lambda r.(p^{(0)}_{\mathcal{R}}, r, p^{(1)}_{\mathcal{R}}(r))$ for some *fixed* $p^{(0)}_{\mathcal{R}}$.

Problem

- $\texttt{verify}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r_1)) \not\Longrightarrow \texttt{verify}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r_2))$ for $r_1 \neq r_2$:

First attempt

A first local hunch...

$$\textsc{L.Extract}\quad \frac{\texttt{verify}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r_1))}{(x, \texttt{extract}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(\texttt{resample}(r_1)), \mathbf{p}_{\mathcal{R}}(\texttt{resample}(r_1)))) \in \mathcal{R}}$$

where $\mathbf{p}_{\mathcal{R}} := \lambda r.(p^{(0)}_{\mathcal{R}}, r, p^{(1)}_{\mathcal{R}}(r))$ for some *fixed* $p^{(0)}_{\mathcal{R}}$.

Problem

- $\texttt{verify}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r_1)) \not\Longrightarrow \texttt{verify}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r_2))$ for $r_1 \neq r_2$:

First attempt

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$$\textsc{L.Extract}\quad \frac{\texttt{verify}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r_1))}{(x, \texttt{extract}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(\texttt{resample}(r_1)), \mathbf{p}_{\mathcal{R}}(\texttt{resample}(r_1)))) \in \mathcal{R}}$$

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Problem

- $\texttt{verify}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r_1)) \not\Longrightarrow \texttt{verify}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r_2))$ for $r_1 \neq r_2$:
- If ϕ is locally true, it says nothing about the distribution of $\left[\ \rho \in \Omega \mid [\![\phi]\!]_\rho\ \right]$.
- Thus, we need to characterize events which holds with non-negligible probability.

An addition to the CCSA logic: ${}_{e}[\_]$ predicate

${}_{e}[\_]$ predicate

For a formula ϕ : **bool** and a non-negligible term e : **real** [non-negl], we define:

$${}_{e}[\phi] \iff \mathsf{Pr}_{\rho\in\Omega}\left(\llbracket\phi\rrbracket_{\rho}\right) \geqslant e$$

- We want the following equivalence:

$$\tilde{\neg}\ [\neg\,\phi]\ \tilde{\Leftrightarrow}\ \tilde{\exists}\, e : \textbf{real}\ [\text{non-negl}].\ {}_{e}[\phi]$$

- and we want

$${}_{e}[\phi(r)]\ \tilde{\to}\ [\phi(\mathtt{resample}(r))]$$

- e : **real** [non-negl] means that $\eta \longmapsto \llbracket e\rrbracket^{\eta}$ is non-negligible,
  i.e. their exists a polynomial P such that: $\exists\,\eta_0 \in \mathbb{N}^*, \forall\,\eta > \eta_0, \llbracket e\rrbracket^{\eta} \geqslant \dfrac{1}{P(\eta)}$.

Are we done yet?

$$\frac{\textsc{G.Extract} \qquad {}_{e}\left[\mathtt{verify}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r))\right]}{\left[(x, \mathtt{extract}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(\mathtt{resample}(r)), \mathbf{p}_{\mathcal{R}}(\mathtt{resample}(r)))) \in \mathcal{R}\right]}$$

where $\mathbf{p}_{\mathcal{R}} := \lambda r.(p_{\mathcal{R}}^{(0)}, r, p_{\mathcal{R}}^{(1)}(r))$ for some *fixed* $p_{\mathcal{R}}^{(0)}$.

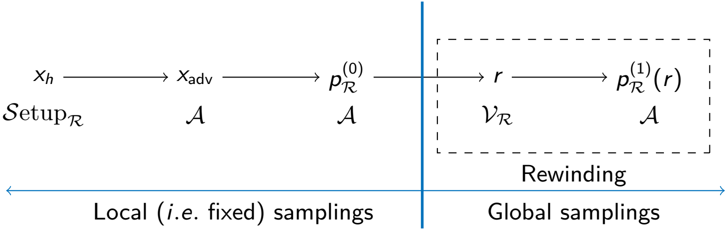
Are we done yet?

G.Extract

$$\frac{{}_{e}\left[\mathtt{verify}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(r))\right]}{\left[(x, \mathtt{extract}_{\mathcal{R}}(x, \mathbf{p}_{\mathcal{R}}(\mathtt{resample}(r)), \mathbf{p}_{\mathcal{R}}(\mathtt{resample}(r)))) \in \mathcal{R}\right]}$$

where $\mathbf{p}_{\mathcal{R}} := \lambda r.(p_{\mathcal{R}}^{(0)}, r, p_{\mathcal{R}}^{(1)}(r))$ for some *fixed* $p_{\mathcal{R}}^{(0)}$.

No, not yet

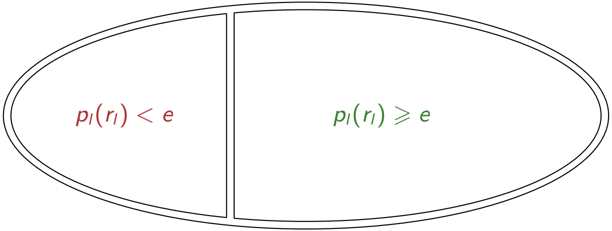


What is missing

- Let $\phi : (r_l, r_g) \longmapsto \phi(r_l, r_g)$ where r_g is the resampled value and r_l refers to other fixed samples.
- We want to study the set $\left\{ \; r_l \;\middle|\; \phi(r_l, r_g) \text{ holds with non-negligible probability on } r_g \; \right\}$.
- Let p_l be the following function

$$p_l := r_l \longmapsto \Pr_{r_g}\Big(\phi(r_l, r_g)\Big)$$

Sampling space (on r_l)



Another addition to the CCSA logic

Selection of sampling space predicate

- Let $\phi : (r_l, r_g) \longmapsto \phi(r_l, r_g)$ be a function predicate.
- Variable r_g is the parameters we want to rewind in the predicate ϕ.
- select is a local predicate saying that locally we are in the "good" case where ϕ holds.

select predicate

$$[\![\text{select}(e, \phi(r_l))]\!]_\rho := \Pr_{r_g}\left([\![\phi(r_l)]\!]_\rho(r_g) \right) \geqslant e.$$

Proof strategy - Step 1

Goal proof under select **guard - Axiomatization**

The G.Extract rule becomes

$$\text{G.SEL-INTRO} \quad \left[\mathsf{select}(e, \psi_{\mathcal{R}}(r_l)) \rightarrow (x(r_l), \mathtt{extract}_{\mathcal{R}}(x(r_l), \mathbf{p}^{(1)}_{\mathcal{R}}(r_l, \mathtt{resample}(r_g)), \mathbf{p}^{(2)}_{\mathcal{R}}(r_l, \mathtt{resample}(r_g))))\right]$$

Where $\psi_{\mathcal{R}}(r_l) := r_g \longmapsto \mathtt{verify}_{\mathcal{R}}(x(r_l), (p^0_{\mathcal{R}}(r_l), r_g, p^1_{\mathcal{R}}(r_g)))$.

Rewinding lemma

Statement

`resample` predicate

Let $\phi : r_g \longmapsto \phi(r_g)$ be a predicate. If $\mathbf{r}_g : \mathbf{nat} \to \tau_g$ then

$$\tilde{\exists}\ k : \mathbf{nat}\ [\text{poly}].\ \tilde{\exists}\ \texttt{resample} : \mathbf{list} \to \tau_g.$$
$$\left[\text{select}(e, \phi) \to \phi(\texttt{resample}(\mathbf{r}_g\, 1, \ldots, \mathbf{r}_g\, k))\right]$$

Proof strategy - Step 2

Glue splitted parts back together

$\mathcal{H} : r \longmapsto \mathcal{H}\ r$ (Hypothesis predicate); Goal : $r \longmapsto$ Goal r (Goal predicate).

$$\dfrac{\textsc{G.Sel-Elim} \quad \tilde{\forall}\ e : \textbf{real}\ [\text{non-negl}].\ \left[\text{select}(e, \mathcal{H}) \to \mathcal{H}\ r \to \text{Goal}\ r\right]}{\left[\mathcal{H}\ r \to \text{Goal}\ r\right]}$$

Proof strategy - Step 2

Glue splitted parts back together

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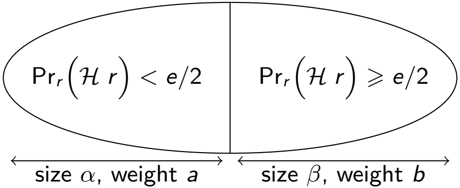
Why does it work?

Proof by contraposition: we want to prove

$$\frac{{}_{e}\left[\mathcal{H}\ r \wedge \neg\,\text{Goal}\ r\right]}{{}_{e/2}\left[\text{select}\left(\frac{e}{2}, \mathcal{H}\right) \wedge \mathcal{H}\ r \wedge \neg\,\text{Goal}\ r\right]}$$



We have $a \leqslant e/2$ and $b \leqslant \beta$.
Therefore, as $a + b \geqslant e$, $\beta \geqslant e/2$

Conclusion

Take aways

- To axiomatize rewinding argument, we have to resample only a part of the random tape;
- We need to talk about formulas sometimes true;
- High-order logic was needed for the rewinding lemma!

Other works done

- Complete formal proof of the permutation secrecy property;
- First complete proof of Terelius & Wikström mixnet protocol.

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What next?

- Reprogrammable Random Oracle Model
- Sigma-protocols → NIZK proof (Fiat-Shamir transform) ...
- ... Towards proof of in practice used mix-network protocols (CHVote and Belenios).

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Thank you for your attention![1]

[1] Icons comes from the Flaticons website (https://www.flaticon.com/)